

TWO-PHASE FLOWS OF INCOMPRESSIBLE CONDENSED MEDIA AND GAS

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A model of weakly compressible media based on the assumption of small volume concentration of gas is developed for the description of flows of powder mixtures. A strong shock-wave approximation is used to describe the dynamics of strong discontinuities. Various approximate formulations of the problem are deduced from a variational principle by restricting the class of function within which a critical point of the action functional is sought.

From the equations of motion for mixtures consisting of an incompressible condensed medium and a gas, approximate dynamic equations are derived using a one-velocity approximation with common pressure of the phases (see [1] and bibliography there) and the assumption that the volume concentration of the gas in the mixture is small. Shock-wave propagation in a mixture of an inviscid fluid and a gas is described using an approximation of strong shock waves that coincides with the model for weakly compressible media derived in [2]. For flows without phase transitions, the shock-wave velocity in this approximation is proportional to the velocity of the medium past the front. A comparison with experimental data for metal powders [3] shows that the model of strong shock waves gives a satisfactory approximation for shock-wave velocities comparable with the velocity of sound in a metal.

For one-dimensional motion of a mixture with strong discontinuities, we prove a variational principle that allows us to employ the Galerkin method to formulate various models in which a solution is obtained from a system of ordinary differential equations.

1. EQUATIONS OF MOTION

Notation. We assume that particles of the gas and the condensed medium that form the mixture are points. The condensed medium will be called the first phase, and the gas will be referred to as the second phase.

The volume dV occupied by the mixture is represented as the sum of two volumes: $dV = dV_1 + dV_2$. Here dV_i are Lebesgue measures (volumes) occupied by the carrier medium and the gas. We assume that for every point of the mixture, the following limits exist:

$$\lim_{dV \rightarrow 0} \frac{dV_i}{dV} = \alpha_i, \quad \alpha_1 + \alpha_2 = 1.$$

The quantities α_1 and α_2 are called the *volume concentrations* of the corresponding phases.

We also assume that the functions ρ_i , p_i , T_i , C_{iv} , and \varkappa_i are defined at every point of the flow domain and represent the density, pressure, temperature, specific heat at constant volume, and the thermal conductivity for the condensed medium ($i = 1$) and the gas ($i = 2$), and the function $\sigma'_{ij} = -p_i \delta_{ij} + \tau'_{ij}$ (δ_{ij} is the Kronecker symbol) is the stress tensor of the condensed medium. The mass of the i th component of the

mixture contained in the volume dV is equal to $\rho_i dV_i$. The density of the mixture is given by the equality $\rho dV = \rho_1 dV_1 + \rho_2 dV_2$, $\rho = \alpha_1 \rho_1 + \alpha_2 \rho_2$. The quantities

$$\beta_i = \frac{\rho_i dV_i}{\rho dV} = \frac{\alpha_i \rho_i}{\rho}, \quad \beta_1 + \beta_2 = 1 \quad (1.1)$$

are called the *mass concentrations* of the mixture components.

Remark 1. Since the mass of each phase in the moving volumes is conserved, the mass concentrations in the particles are constant.

Next, it is assumed that the equation of state for the gas has the form

$$p_2 = f(\rho_2, T_2). \quad (1.2)$$

In particular, for a perfect gas, $p_2 = \rho_2 R T_2$. The condensed medium is considered incompressible: $\rho_1 = \text{const}$. We note that for specific volumes $dv_i = 1/\rho_i$ and $v = 1/\rho$, the following equalities are valid:

$$v = \beta_1 v_1 + \beta_2 v_2, \quad \alpha_1 v = \beta_1 v_1, \quad \alpha_2 v = \beta_2 v_2 = v - \beta_1 v_1. \quad (1.3)$$

Stress Tensor and Thermal Conductivity. To determine the stress tensor of the mixture $\sigma_{ij} = -p_i \delta_{ij} + \tau_{ij}$, we consider a surface element $dS = dS_1 + dS_2$ (dS_i is the surface measure of i th phase) with the normal \mathbf{n} and set

$$\left(-pn_i + \sum_{j=1}^3 \tau_{ij} n_j \right) dS = -(p_1 dS_1 + p_2 dS_2) + \sum_{j=1}^3 \tau'_{ij} n_j dS_1,$$

where $i = 1, 2$, and 3 . Arbitrarily extending the surface element to a cylinder of height dl and taking into account that $dS_i/dS = dV_i/dV = \alpha_i$, from the preceding equality we obtain $p = \alpha_1 p_1 + \alpha_2 p_2$ and $\tau_{ij} = \alpha_1 \tau'_{ij}$. In other words, the volume-averaged and surface-averaged values coincide [1].

In what follows, we assume that in a volume element the pressures in each phase coincide: $p_1 = p_2 = p$. The stress tensor of the mixture has the form

$$\sigma_{ij} = -p \sigma_{ij} + \alpha_1 \tau'_{ij} = -p \delta_{ij} + \beta_1 \rho \tau'_{ij} / \rho_1. \quad (1.4)$$

The thermal conductivity of the mixture is defined similarly:

$$\varkappa = \alpha_1 \varkappa_1 + \alpha_2 \varkappa_2 = (\beta_1 \rho / \rho_1) \varkappa_1 + \beta_2 (1 - \beta_1 \rho / \rho_1) \varkappa_2. \quad (1.5)$$

Internal Energy and Entropy. The internal energy of the mixture is defined by the equality $\varepsilon = \beta_1 \varepsilon_1 + \beta_2 \varepsilon_2$, where $\varepsilon_i = C_{vi} T_i$. According to the first law of thermodynamics $dQ_1 + dQ_2 = d(\beta_1 \varepsilon_1) + d(\beta_2 \varepsilon_2) + p dv$, where dQ_i is the increase in quantity of heat for each component per unit mass of the mixture. With allowance for Remark 1 and Eq. (1.3), the last relation can be written as

$$dQ_1 + dQ_2 = \beta_1 d\varepsilon_1 + \beta_2 d\varepsilon_2 + \beta_2 p dv_2. \quad (1.6)$$

The flow of the mixture depends greatly on the heat-transfer processes. Below we consider two models for describing two-phase media.

Case A: Heat-Non-Conducting Condensed Medium and Gas. Since there is no heat exchange between the phases, then, according to (1.6), the following equalities must hold: $dQ_1 = \beta_1 d\varepsilon_1$, $dQ_2 = \beta_2 (d\varepsilon_2 + p dv_2) = \beta_2 T_2 ds_2$, where s_2 is the entropy of the second phase.

Remark 2. If external sources of heat are absent, the change in the internal energy of the first phase is equal to the work done by the forces of internal friction.

We define the entropy of the mixture by the equality $s = \beta_2 s_2$. For a polytropic gas,

$$p/p_0 = A(s, \beta_2) (\rho_2 / \rho_{20})^\gamma. \quad (1.7)$$

Remark 3. If external sources of heat are absent, then for continuous flows, the entropy in the particles is conserved and $A(s, \beta_2) = 1$.

Case B: Heat-Conducting Condensed Medium and Gas. In this case, we assume that in any volume element, the temperatures of both phases are identical. The equality $C_v \rho dV = C_{v1} \rho_1 dV_1 + C_{v2} \rho_2 dV_2$ defines the specific heat of the mixture:

$$C_v = \beta_1 C_{v1} + \beta_2 C_{v2}. \quad (1.8)$$

By definition, $d\varepsilon_i = C_{vi} dT$ and $dQ = dQ_1 + dQ_2 = C_v dT + \beta_2 p dv_2 = \beta_1 C_{v1} dT + \beta_2 T ds_2$. Therefore, the entropy of the mixture is given by $s = \beta_1 C_{v1} \ln T + \beta_2 s_2$.

As in the case of heat-non-conducting phases, for a polytropic gas,

$$p/p_0 = B(s, \beta_2)(\rho_2/\rho_{20})^{\gamma_*}, \quad (1.9)$$

where

$$\gamma_* = 1 + \beta_2 R/C_v. \quad (1.10)$$

Remark 4. In contrast to heat-non-conducting media, the entropy of the mixture increases ($ds > 0$) in the presence of internal friction: $B(s, \beta_2) = 1$, if the entropy in the particles is constant.

Laws of Conservation of Specific Concentration, Mass, and Momentum. The above reasoning shows that the mixture of an incompressible condensed medium and a gas can be treated as a condensed medium with velocity field \mathbf{u} , density ρ , and the stress tensor σ_{ij} defined by Eq. (1.4). According to Remark 1,

$$\frac{d\beta_2}{dt} = 0, \quad (1.11)$$

where the operator $d/dt = \partial/\partial t + (\mathbf{u} \cdot \nabla)$ is the total derivative with respect to time.

For flows without strong discontinuities, the laws of conservation of mass and momentum have the form

$$\frac{\partial \rho}{\partial t} + \rho \operatorname{div} \mathbf{u} = 0; \quad (1.12)$$

$$\frac{d\mathbf{u}}{dt} + \frac{1}{\rho} \nabla p = \frac{1}{\rho \rho_1} \operatorname{div} (\beta_1 \rho \boldsymbol{\tau}) + \mathbf{f}, \quad (1.13)$$

where \mathbf{f} is the body force, $\boldsymbol{\tau}$ is the stress tensor deviator, and, by definition,

$$(\operatorname{div} (\beta_1 \rho \boldsymbol{\tau}))_i = \sum_{j=1}^3 \frac{\partial}{\partial x_j} (\beta_i \rho \tau_{ij}).$$

Laws of Conservation of Energy. It is known that the total change in the kinetic energy of the medium results from the work done by body forces and surface forces in a volume element. The change in the internal energy ε is related to the work done by friction forces on changing the volume of the medium and to heat-transfer processes:

$$\rho \frac{d\varepsilon}{dt} + p \nabla \cdot \mathbf{u} - \alpha_2 G = \nabla \cdot \boldsymbol{\theta}, \quad (1.14)$$

where $\boldsymbol{\theta}$ is the current density of the heat transferred through a unit area of isothermal surface per unit time

and $G = \sum_{i,j=1}^3 \tau_{ij} \frac{\partial u_i}{\partial x_j}$ [τ^* is the dissipative (viscous) component of the tangential stress tensor].

The internal energy varies differently, depending on the heat-conduction processes. We consider two cases.

Case A: Heat-Non-Conducting Condensed Medium and Gas. According to Remark 2, the work done by the forces of internal friction is equal to the change in the internal energy of the first phase. Therefore, Eq. (1.14) splits into two equations:

$$\rho \frac{d}{dt} (\beta_1 \varepsilon_1) - \alpha_1 G = 0, \quad \rho \frac{d}{dt} (\beta_2 \varepsilon_2) + p \nabla \cdot \mathbf{u} = 0.$$

Taking into account the law of conservation of mass concentration (1.11) and the definitions of the specific internal energies $\varepsilon_i = C_{vi} T_i$, and (1.1), we obtain

$$C_{v1} \frac{dT_1}{dt} - \frac{\rho}{\rho_1} G = 0; \quad (1.15)$$

$$\beta_2 C_{v2} \rho \frac{dT_2}{dt} + p \nabla \cdot \mathbf{u} = 0. \quad (1.16)$$

Case B: Heat-Conducting Condensed Medium and Gas [1]. In this case, equalities (1.5) and (1.8) define the thermal conductivity and the specific heat at constant volume of the mixture. Letting $\varepsilon = C_v T$, from (1.14) we obtain the following equation for the temperature:

$$C_v \rho \frac{dT}{dt} + p \nabla \cdot \mathbf{u} - \alpha_2 G = \nabla \cdot (\kappa \nabla T). \quad (1.17)$$

In view of (1.3), the equation of state of the mixture (2.2) can be written as

$$p = g(\beta_2, \rho, T), \quad (1.18)$$

where $T = T_2$ for heat-non-conducting phases.

Remark 5. If the entropy in the particles is conserved, then for a polytropic mixture, Eqs. (1.3), (1.7), and (1.9) lead to the law of conservation of entropy:

$$\frac{d}{dt} [(v - \beta_1 v_1)^k p] = 0, \quad (1.19)$$

where $k = \gamma$ for the case A and $k = \gamma_*$ for the case B. The adiabatic exponent is defined by formula (1.10).

We note that in the case of heat-conducting phases, the flow is isentropic if the stress tensor is spherical and the thermal conductivity is negligible.

Remark 6. Equations (1.11)–(1.13), (1.15), and (1.16) and the equation of state (1.18) represent a complete system of equations for the mass concentration, velocity field, pressure, and temperature of each mixture component in heat-non-conducting phases.

Equations (1.11)–(1.13) and (1.17) and the equation of state (1.18) are a complete system that defines the flow parameters in the case of equal temperatures of the phases.

Remark 7. For isentropic flows, the mass concentration, the density, and the velocity field are obtained from Eqs. (1.11)–(1.13), the equation of state (1.18), and the law of conservation of entropy (1.19).

Remark 8. For isentropic flows, the velocity of sound in the mixture is calculated from the formula (see also [1])

$$c = \left(\frac{\partial p}{\partial \rho} \right)^{1/2} = \left(\frac{k p}{\rho_2} \frac{\partial \rho_2}{\partial \rho} \right)^{1/2} = \left(\frac{k \rho_2}{\gamma \alpha_2 \rho} \right)^{1/2} c_0,$$

where c_0 is the local velocity of sound in the gas.

For heat-non-conducting phases ($k = \gamma$) with specified ρ_2 , the velocity of sound in the mixture $c(\alpha_2)$ is equal to the velocity of sound in the gas c_0 if $\alpha_2 = 1$ and $\alpha_2 = \alpha_2^*$, $c(\alpha_2) < c_0$ if $\alpha_2^* < \alpha_2 < 1$, and $c(\alpha_2) > c_0$ if $\alpha_2 < \alpha_2^*$. Here $\alpha_2^* = (\rho_1 - |\rho_1 - 2\rho_2|)/(2(\rho_1 - \rho_2))$. The minimum velocity of sound in the mixture

$$c_* = 2 \left[\frac{\rho_2}{\rho_1} \left(1 - \frac{\rho_2}{\rho_1} \right) \right]^{1/2} c_0$$

is attained for $\alpha_2 = \rho_1/(2(\rho_1 - \rho_2))$ ($\rho_2 < \rho_1$). If $\rho_2/\rho_1 \ll 1$, then $c_* \simeq 2\sqrt{\rho_2/\rho_1} c_0 \ll c_0$.

Initial and Boundary Conditions. To single out a unique solution of the equations describing the dynamics of mixtures, it is necessary to specify initial and boundary conditions. We assume that at $t = 0$, the velocity field \mathbf{u}_0 , the mass concentration β_2 (or β_1), and the density of the gas ρ_{20} are known. The volume concentration and the density of the mixture are obtained from the formulas $\alpha_2 = \beta_2 \rho_1 / (\beta_2 \rho_1 + \beta_1 \rho_2)$ and $\rho = \rho_1 \rho_2 / (\beta_2 \rho_1 + \beta_1 \rho_2)$. In addition, the temperatures T_{10} and T_{20} of each phase must be given. The pressure is determined from the equation of state (1.18).

On the rigid walls, the condition of no flow through the boundary must be satisfied: the normal velocity of the particles adjacent to the wall must coincide with the normal velocity of the wall. This condition suffices if the first phase is an inviscid incompressible fluid. For other condensed media, additional boundary conditions appear. If the condensed medium is a viscous incompressible fluid, the no-slip condition must be satisfied on the rigid walls: the velocity of a fluid particle on the wall is equal to the velocity of the wall. For a plastic

medium, the friction law must be satisfied on the wall (for example, the Amanton–Coulomb law: the density of the friction force on the rigid wall is proportional to the normal stress and its direction is opposite to the motion of particles of the medium).

On the free boundaries, the kinematic condition (condition of no normal flow) and the dynamic conditions (the normal stress coincides with the external pressure and tangential stress is absent) must be satisfied. For a heat-conducting mixture, in addition, conditions for temperature must be specified on the rigid walls and free boundaries.

Shock Waves. If the first phase is an inviscid incompressible fluid (spherical stress tensor), flows with strong discontinuities are possible. At a strong discontinuity, the equality of the mass concentrations, mass flux, and momentum must be satisfied:

$$[\beta_2]_D = 0, \quad [\rho(D - u)]_D = 0, \quad [\rho(D - u)^2 + p]_D = 0. \quad (1.20)$$

Here the square brackets denote an operator that determines the jump of the corresponding function at the shock wave, D is the velocity of the shock-wave propagation, and u is the velocity of the mixture normal to the shock-wave front.

For a perfect gas, the specific energy can be expressed as

$$u^2/2 + pv + C_v T = u^2/2 + pv + (C_v/R)pv_2.$$

Here and below, we assume that $C_v = \beta_2 C_{v2}$ and $T = T_2$ in the case A and $C_v = \beta_1 C_{v1} + \beta_2 C_{v2}$ in the case B; the specific volume v is defined by Eq. (1.3).

The law of conservation of energy at the shock wave is written as

$$\left[\frac{1}{2}(D - u)^2 + pv + \frac{C_v}{\beta_2 R}(v - \beta_1 v_1)p \right]_D = -ve. \quad (1.21)$$

Here $1/v$ is the density of the mixture past the shock-wave front, e is the energy loss (due to some factors, for example, chemical processes) per unit volume of the mixture after passage through the shock wave.

From the jump conditions (1.20) and (1.21), we deduce a shock-adiabat equation. For heat-non-conducting media and a perfect gas, it has the form

$$-\frac{1}{2}(v_0 - v)p + \frac{v - \beta_1 v_1}{\gamma - 1}p = \frac{1}{2}(v_0 - v)p_0 + \frac{v_0 - \beta_1 v_1}{\gamma - 1}p_0 - ev. \quad (1.22)$$

2. APPROXIMATE FORMULATIONS

Approximation Assuming Small Volume Concentration of the Gas in the Mixture. We assume that, at the initial time, the mass concentration of the gas in the mixture $\beta_2 = \delta^2$ is constant and $\delta \ll 1$. Since the mass concentrations in a particle are conserved, the value δ remains constant for subsequent times.

Assuming that the components of the mixture are not heat-conducting, we set $v = v_1(1 - \delta^2 + \delta^2\theta)$, $\mathbf{u} = \delta\mathbf{u}'$, $t = \delta t'$, $\tau_{ij} = \delta^2\tau'_{ij}$, $G = \delta^{2\sigma+1}G'$, $T_1 = \delta^{2\sigma}T'_1$, $T_2 = T'_2$, $e_0 = \delta^2e'_2$, and $e_1 = \delta^2e'_1$. Here $\sigma = 0$ or $\sigma = 1$, depending on the properties of the first phase.

Assuming that the gas is perfect and using the equation of state $pv_2 = RT_2$, (1.3) and the laws of conservation (1.12), (1.13), (1.15), and (1.16), with accuracy up to terms of the second order in δ , we obtain (primes are omitted):

$$\theta p = \rho_1 RT_2, \quad \frac{\partial \theta}{\partial t} - \nabla \cdot \mathbf{u} = 0; \quad (2.1)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{\rho_1} \nabla p = \frac{\delta^{2\sigma}}{\rho_1} \mathbf{div} \tau + \mathbf{f}; \quad (2.2)$$

$$C_{v2}\rho_1 \frac{\partial T_2}{\partial t} + p \nabla \cdot \mathbf{u} = 0; \quad (2.3)$$

$$C_{v1} \frac{\partial T_1}{\partial t} - G = 0. \quad (2.4)$$

Equations (2.1)–(2.3) together with initial data and boundary conditions are used to obtain the velocity field, the pressure field, the temperature, and the specific volume θ . Equality (2.4) defines the temperature of the first phase.

Remark 9. In the case of a viscous fluid, the stress tensor deviator depends linearly on the derivatives of the velocity with respect to spatial variables, and therefore, $\sigma = 1$. On the right side of (2.2), the term proportional to $\delta^{2\sigma}$ is retained in order to take into account the smoothing properties of viscosity and avoid difficulties in formulating the boundary conditions.

If tangential stresses are absent (spherical tensor), then, according to Remark 3, the flow is isentropic and system (2.1)–(2.4) is considerably simplified. To describe the flow, it suffices to use the law of conservation of mass [the second equation in (2.1)], the law of conservation of entropy (1.19), and the law of conservation of momentum (without body forces). Within the framework of the given approximation, we obtain

$$\frac{\partial \theta}{\partial t} - \nabla \cdot \mathbf{u} = 0, \quad \frac{\partial}{\partial t}(p\theta^\gamma) = 0; \quad (2.5)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{\rho_1} \nabla p = 0. \quad (2.6)$$

According to (2.4), the temperature of the fluid is constant. The temperature of the gas in the mixture is determined from the equation of state $\theta p = \rho_1 R T_2$. Relation (2.3) is a consequence of the equation of state and formulas (2.5) and (2.6).

In flows without tangential stresses (the first phase is an inviscid incompressible fluid), shock waves are possible. At a jump, the conditions of equality of mass and momentum fluxes (1.20) are satisfied with accuracy to terms of lower orders in δ if

$$(\theta_0 - \theta)V = u - u_0, \quad p - p_0 = \rho_1(u - u_0)V, \quad (2.7)$$

where $V = \delta^{-1}D$.

Within the framework of this approximation, from the shock-adiabat equation (1.22) we obtain

$$((\gamma + 1)\theta - (\gamma - 1)\theta_0)p = ((\gamma + 1)\theta_0 - (\gamma - 1)\theta)p_0 - (\gamma - 1)e/2. \quad (2.8)$$

Using relations (2.7), we eliminate the functions p and θ from the shock-adiabat equation and obtain the following relation between the shock-wave velocity V and the velocity of the gas u :

$$2\theta_0\rho_1uV^2 - \left((\gamma + 1)\rho_1u^2 - \frac{\gamma - 1}{2}e\right)V - 2\gamma p_0u = 0. \quad (2.9)$$

Converting to the initial (without primes) variables and taking into account that $\alpha_2 = \delta^2\theta$, we obtain

$$2\alpha_{20}uD^2 - \left((\gamma + 1)u^2 - \frac{\gamma - 1}{2\rho_1}e\right)D - 2\frac{\rho_{20}}{\rho_1}c_0^2 = 0, \quad (2.10)$$

where $c_0^2 = \gamma p_0/\rho_{20}$ is the square of the velocity of sound in the gas. For $e = 0$ (no heat exchange between the phases), we have

$$D = \frac{\gamma + 1}{4\alpha_{20}} \left(u + \left(u^2 + \frac{16}{(\gamma + 1)^2} \frac{\rho_{20}}{\rho_1} \alpha_{20} c_0^2 \right)^{1/2} \right).$$

For $\rho_{20}/\rho_1 \ll 1$, the last formula becomes

$$D = (\gamma + 1)u/(2\alpha_{20}). \quad (2.11)$$

If, in addition, $u^2 \gg (\gamma - 1)e/(2(\gamma + 1)\rho_1)$, then from (2.10) we have the approximate relation

$$D = \frac{2}{4\alpha_{20}} \left((\gamma + 1)u - \frac{\gamma - 1}{2\rho_1} \frac{e}{u} \right). \quad (2.12)$$

Remark 10. Within the framework of this approximation, the contact discontinuities in the mixture are fixed.

Approximation of Strong Shock Waves. In accordance with the shock-adiabat equation (2.8), $\theta \rightarrow \theta_0(\gamma - 1)/(\gamma + 1)$ or $\rho_2 \rightarrow \rho_{20}(\gamma + 1)/(\gamma - 1)$ as $p \rightarrow \infty$, because $\rho_2 = \delta^2 \rho_1/\theta$. We assume that past the shock-wave front (in its neighborhood)

$$\frac{p}{p_0} \gg 1, \quad \theta = \frac{\gamma - 1}{\gamma + 1} (\theta_0 - \theta'), \quad \theta' \ll 1.$$

Let L be the characteristic linear dimension. We introduce dimensionless variables by the equalities

$$\begin{aligned} x' &= \frac{x}{L}, & t' &= \frac{1}{L} \sqrt{\frac{p_0}{\rho_1}} t, & p &= \frac{(\gamma - 1)\theta_0^2}{(\gamma + 1)^2\theta} p_0 p', \\ u &= \frac{\theta_0}{\gamma + 1} \sqrt{\frac{p_0}{\rho_1}} u, & V &= \sqrt{\frac{p_0}{\rho_1}} V', & e &= \frac{\theta_0^2}{(\gamma + 1)^2\theta} p_0 e_0. \end{aligned}$$

Taking into account that

$$\theta \simeq \frac{\gamma - 1}{\gamma + 1} \theta_0, \quad p \simeq \frac{\theta_0 p_0}{\gamma + 1} p', \quad \nabla p \simeq \frac{\theta_0 p_0}{L(\gamma + 1)} \nabla' p',$$

from system (2.5), (2.6) we obtain (primes are omitted)

$$p_t + p \nabla \cdot \mathbf{u} = 0, \quad \mathbf{u}_t + \nabla p = 0. \quad (2.13)$$

Letting $p_0 = 0$ in the equations of conservation of momentum in (2.7) and the $(V-u)$ diagram (2.9), as is customary in the theory of strong shock waves, we obtain the following relations at the jump:

$$p = uV, \quad 2uV = u^2 - 2e_0. \quad (2.14)$$

In physical variables, the second equality in (2.14) coincides with formula (2.12).

Remark 11. The law of conservation of mass in (2.7) serves as the boundary condition for the function θ' . Using (2.14), we write it in the form $(\gamma - 1)\theta' = 2e\theta_0/p$.

Remark 12. Conditions (2.14) at the shock wave can be written as

$$p = uV, \quad u^2/2 = p + e_0. \quad (2.15)$$

According to [2], the flow of the mixture can be treated as flow of a weakly compressible polytropic gas (the adiabatic exponent is much larger than unity) with phase transitions at the shock wave (at $e \neq 0$) due to physical processes such as endothermic combustion.

Comparison with Experiment. Bakanova et al. [3] give experimental data for the shock-wave velocity D as a function of the flow velocity of the medium u for powders of molybdenum, tungsten, copper, and aluminum. It follows from the diagrams presented in [3] that this function can be considered linear ($D = au + b$) over a fairly wide range of velocities. Below, we give the equations (from [3]) of $(D - u)$ -diagrams for each powder and the corresponding formulas (2.11) for various values of the porosity m , which is related to the volume concentration by the formula $\alpha_2 = (m - 1)/m$.

For molybdenum powder,

$$\begin{aligned} D_{\text{exp}} &= 2.05u + 0.03 \text{ and } D = 2.66u \text{ for } m = 1.82 \text{ and } 0.58 \leq u \leq 2.31, \\ D_{\text{exp}} &= 1.58u - 0.07 \text{ and } D = 1.77u \text{ for } m = 3.1 \text{ and } 0.64 \leq u \leq 3.21. \end{aligned}$$

For tungsten powder,

$$\begin{aligned} D_{\text{exp}} &= 1.97u + 0.13 \text{ and } D = 2.68u \text{ for } m = 1.81 \text{ and } 0.52 \leq u \leq 1.98, \\ D_{\text{exp}} &= 1.48u - 0.11 \text{ and } D = 1.67u \text{ for } m = 3.55 \text{ and } 0.61 \leq u \leq 2.86. \end{aligned}$$

For copper powder,

$$\begin{aligned} D_{\text{exp}} &= 1.82u + 0.21 \text{ and } D = 2.52u \text{ for } m = 1.91 \text{ and } 0.6 \leq u \leq 2.44, \\ D_{\text{exp}} &= 1.62u + 0.06 \text{ and } D = 1.81u \text{ for } m = 2.98 \text{ and } 0.64 \leq u \leq 2.73. \end{aligned}$$

For aluminum powder,

$$\begin{aligned} D_{\text{exp}} &= 1.85u + 0.2 \text{ and } D = 2.39u \text{ for } m = 2.01 \text{ and } 0.65 \leq u \leq 2.97, \\ D_{\text{exp}} &= 1.59u - 0.05 \text{ and } D = 1.78u \text{ for } m = 3.01 \text{ and } 0.67 \leq u \leq 3.16. \end{aligned}$$

Here D_{exp} is the shock-wave velocity obtained experimentally and D is the corresponding velocity obtained from (2.11). Experimental ($D-u$) diagrams were plotted using the outermost points of the specified ranges of velocities. The unit of measurement for the velocity is 1 km/sec.

These data show that the approximation of strong shock waves is adequate for describing the dynamics of powder media with shock-wave velocities comparable with the velocity of sound in a metal.

Remark 13. The one-dimensional system (2.13) $p_t + pu_x = 0$, $u_t + px = 0$ is linearized if $\alpha = u$ and $\beta = 2\sqrt{p}$ are chosen as independent variables, and $t = t(\alpha, \beta)$ and $x = x(\alpha, \beta)$ are chosen as unknown quantities: $x_\alpha + \beta t_\beta/2 = 0$ and $x_\beta + \beta t_\alpha/2 = 0$. According to (2.15), in the plane (α, β) , the shock-wave equation has the form $|\alpha| = \sqrt{2(\beta^2 + 4e_0)}/2$ or $\beta = \sqrt{2\alpha^2 - 4e_0}$. At the shock, the following condition must be satisfied:

$$\frac{x_\alpha + \beta_\alpha x_\beta}{t_\alpha + \beta_\alpha t_\beta} = \frac{\beta^2}{4\alpha}.$$

If we introduce the stream function $x = \beta\psi_\beta/2$, $t = -\psi_\alpha$, it satisfies the equation of the membrane: $\psi_{\alpha\alpha} - (\beta\psi_\beta)_\beta/\beta = 0$.

3. PULSED COMPRESSION OF A FINITE MASS OF A MIXTURE

Formulation of the Problem. Let, at the initial time $t = 0$, a motionless mixture occupy the volume $0 \leq r \leq 1$. We assume that a shock wave propagates from the boundary $r = 1$ to the center of symmetry $r = 0$ and the position of the front is given by the equalities $r = 1 - R(t)$ and $R(0) = 0$. We assume that the flow past the shock-wave front is described by the one-dimensional system (2.13) with boundary conditions (2.15). To describe the motion of the medium taking into account the characteristic features of the problem, it is reasonable to introduce the new independent variables $\tau = R(t)$, and $\xi = (1 - r)/\tau$ and let $R'(t) = q(\tau)/\sqrt{\tau}$, $u(r, t) = -q(\tau)V(\xi, \tau)/\sqrt{\tau}$, and $p(r, t) = P(\xi, \tau)/\tau$. In the new variables, the one-dimensional system (2.10) becomes

$$\tau \frac{\partial \ln P}{\partial \tau} - \xi \frac{\partial \ln P}{\partial \xi} + r^{-\nu} \frac{\partial}{\partial \xi} (r^\nu V) = 1; \quad (3.1)$$

$$\tau q \frac{\partial}{\partial \tau} (qV) - q^2 \xi \frac{\partial}{\partial \xi} V - \frac{q^2}{2} V + \frac{\partial}{\partial \xi} P = 0, \quad (3.2)$$

where $r = 1 - \tau\xi$, and the geometric parameter $\nu = 0, 1$, and 2 for plane, cylindrical, and spherical waves, respectively.

At the shock wave, the following jump relations must be satisfied:

$$q^2 V = P, \quad q^2 V^2/2 = P + e_0 \tau \quad (\xi = 1). \quad (3.3)$$

The boundary $r = 1$ is treated as a fixed rigid wall:

$$V = 0 \quad (\xi = 0), \quad (3.4)$$

or the interface between the medium and vacuum:

$$P = 0 \quad (\xi = 0). \quad (3.5)$$

The flow of the medium depends strongly on the method of initiation of the shock wave. Therefore, in the domain

$$\Pi = \{(\xi, \tau): 0 < \xi < 1, \tau_0 < \tau < \tau_1\},$$

with $0 < \tau_0 < \tau_1 < 1$, we consider the family of problems (3.1)–(3.4) or (3.1)–(3.3), and (3.5) with the initial data

$$q = q_0, \quad V = V_0(\xi), \quad P = P_0(\xi) \quad (\tau = \tau_0). \quad (3.6)$$

The initial data must be compatible with boundary conditions (3.3)–(3.5):

$$q_0^2 V_0(1) = P_0(1), \quad q^2 V_0^2/2 = P_0 + \epsilon_0 \tau_0; \quad (3.7)$$

$$V_0(0) = 0 \quad \text{or} \quad P_0(0) = 0. \quad (3.8)$$

The compatibility conditions (3.7) define the initial data for the function $q(\tau)$.

For the known function $q(\tau)$, the law of motion for the shock-wave front is determined by solving the Cauchy problem

$$\sqrt{R(t)} R'(t) = q(R(t)), \quad R(t_0) = \tau_0. \quad (3.9)$$

The aim of the further investigation is to determine the class of initial data (3.6) for which it is possible to write approximate solutions of the problems formulated above in the limit $\tau_0 \rightarrow 0$.

Variational Formulation. System (3.1)–(3.3) with boundary conditions (3.4) or (3.5) admits the energy integral

$$\int_0^1 r^\nu \left(\frac{q^2}{2} V^2 + P + \epsilon_0 \tau \right) d\xi = E \quad (3.10)$$

(this can be verified by differentiation with respect to τ) with the constant E calculated from the initial data (3.6).

Variational Principle. The solutions of the equations of motion for a continuous medium (3.1)–(3.3) and (3.4) or (3.5) and (3.6) in the domain Π coincide with the extremals of the action functional

$$S(q, V, P) = \int_{\tau_0}^{\tau_1} L d\tau$$

with the Lagrangian

$$L = \frac{\sqrt{\tau}}{q} \int_0^1 \left(\frac{q^2}{2} V^2 - P - \epsilon_0 \tau \right) r^\nu d\xi + \frac{\sqrt{\tau}}{q} E$$

in the class of functions satisfying the initial data (3.6), the constraint equation (3.1), and the additional constraint $V = 0$ for $\xi = 0$ in the case of boundary condition (3.4).

Indeed, variation of the functional S with respect to q yields the energy integral. Let

$$P = \exp(-r^{-\nu} g_\xi), \quad V = r^{-\nu} \left[(\tau g)_\tau - \xi g_\xi + \frac{1 - r^{\nu+1}}{\tau(\nu+1)} \right], \quad (3.11)$$

so that the continuity equation (3.1) is automatically satisfied. The initial data (3.6) will be satisfied if for $\tau = \tau_0$ we set

$$g = g_0(\xi), \quad \tau_0 g_\tau = r_0^\nu V_0 + \xi g'_0 - g_0 - \frac{1 - r_0^{\nu+1}}{\tau_0(\nu+1)}, \quad (3.12)$$

where

$$r_0(\xi) = 1 - \tau_0 \xi, \quad g_0(\xi) = \int_\xi^1 r_0^\nu \ln P(\xi) d\xi,$$

and obtain g_0 from the compatibility conditions (3.7). Boundary condition (3.4) will also be satisfied if we put

$$\tau g = \tau_0 g_0 \quad (\xi = 0). \quad (3.13)$$

From representation (3.11) we have

$$\delta P = -r^{-\nu} P \delta g_\xi, \quad \delta V = r^{-\nu} [(\tau g)_\tau - \xi \delta g_\xi]. \quad (3.14)$$

Varying the action functional with respect to V and P , taking into account the representation (3.14), and assuming that $\delta g(\xi, \tau_0) = \delta g(\xi, \tau_1) = 0$, after integration by parts we obtain the following equation in variations:

$$\begin{aligned} & \iint_{\Pi} \frac{\sqrt{\tau}}{q} \left(\sqrt{\tau} \tau g \frac{\partial}{\partial \tau} \frac{qV}{\sqrt{\tau}} - \xi q^2 \frac{\partial V}{\partial \xi} + \frac{\partial P}{\partial \xi} \right) \delta g(\xi, \tau) d\tau d\xi \\ &= - \int_{\tau_0}^{\tau_1} \frac{\sqrt{\tau}}{q} P(0, \tau) \delta g(0, \tau) d\tau + \int_{\tau_0}^{\tau_1} \frac{\sqrt{\tau}}{q} [P(1, \tau) - q^2 V(1, \tau)] \delta g(1, \tau) d\tau. \end{aligned} \quad (3.15)$$

This, in view of the arbitrariness of $\delta g(\xi, \tau)$ and $\delta g(1, \tau)$, results in Eq. (3.2) and the first of the conditions (3.3). If $\delta g(0, \tau)$ is arbitrary, then (3.15) implies boundary condition (3.5) [in the case of boundary condition (3.4), Eq. (3.13) yields $\delta g(0, \tau) = 0$]. The second boundary condition in (3.3) follows from the energy integral (3.10) after differentiation with respect to τ . Thus, the variational principle is proven.

Remark 14. In what follows, we shall seek solutions of problems (3.1)–(3.6) that are regular for $\tau_0 \rightarrow 0$. Therefore, the initial data cannot be arbitrary. From (3.11) with $\tau_0 \rightarrow 0$, we obtain $P_0 = \exp(-g_0 \xi)$ and $V_0 = g_0 - \xi g_0 \xi + \xi$, letting $\tau_0 \rightarrow 0$ in (3.12) we find that $(\xi P_0)' = p_0 V_0'$, and from (3.7) and (3.8) in the limit $\tau_0 \rightarrow 0$ we have $V_0(1) = 2$, $P_0(1) = 2q_0^2$, $V_0(0) = 0$ or $P_0(0) = 0$. The initial value for q is determined from the energy integral in the limit $\tau_0 \rightarrow 0$ using the given constant E .

Approximate Solutions. For short times, the flow can be considered plane-parallel because $r \sim 1$. The constant e_0 can be set equal to zero. For $e_0 = 0$, problem (3.1)–(3.4) has the stationary solution [2]

$$U^{2/3}(U-1)^{-1/2}(3-U)^{5/6} = \xi^{-1} 2^{2/3}, \quad P = \frac{q^2}{2} \frac{\xi^2 U^2}{U-1}, \quad q^2 = \frac{3}{4} E,$$

where $U = V/\xi$. The law of motion for the shock-wave front is defined by the equality $R(t) = 3 \cdot 4^{-2/3} E^{1/3} t^{2/3} \simeq 1.19 E^{1/3} t^{2/3}$. We note that $P(1) = 3E/2$ and $P(0) = (3E/4) 4^{2/3}$. Approximate solutions of problems (3.1)–(3.4) with initial data of the form (3.6) are also given in [2].

Equations (3.1) and (3.2) have no stationary solutions satisfying boundary condition (3.5) and boundary condition (3.3) (for $e_0 = 0$). Therefore, hereinafter we construct approximate solutions that give averaged values of the velocity and the pressure at sections $\tau = \text{const}$. We consider a family of functions $g(\xi, \mu)$ that depend on the vector parameter $\mu = (\mu_1, \dots, \mu_k)$, and, in accordance with (3.11), we set $P(\xi, \mu) = \exp(-q\xi)$ and $V(\xi, \mu) = g - \xi q\xi + \xi$. With this definition of P and V , the equation of continuity (3.1) is automatically satisfied. In addition, we assume that

$$g(0, \mu) = V(0, \mu) = 0 \quad \text{or} \quad P(0, \mu) = 0.$$

The variations $\delta g = \nabla_{\mu} g \cdot \delta \mu$ of the functions g are not finite in time. Therefore, we consider a sequence of variations of the form $\delta g_n = \varphi_n(\tau) \delta g(\xi)$, where $\varphi_n(\tau)$ are smooth functions that vanish at $\tau = \tau_0$ and $\tau = \tau_1$ and $\varphi_n(\tau) \rightarrow 1$ for $\tau_0 < \tau < \tau_1$. Substituting δg_n into formula (3.15) and assuming that the function q is constant, in the limit $n \rightarrow \infty$, after integration by parts, we obtain the equality

$$\frac{3}{2} q^2 \int V \nabla_{\mu} g d\xi = \int_0^1 (q^2 \xi V - P) \nabla_{\mu} g d\xi, \quad (3.16)$$

which is valid due to the arbitrariness of $\delta \mu$. Thus, we have the following approximate formulation of the problem: *it is required to find the parameters μ and q from system (3.16) and the energy integral (3.10).*

In the simplest case $V = \mu_1 \xi$ and $P = \mu_2 \xi^{\mu_1 - 1}$, the corresponding solution has the form [2] $\mu_1 = 3/2$, $\mu_2 = E$, and $q_0^2 = 8E/9$. The law of motion for the shock-wave front is given by the formula $R(t) = 2^{1/3} E^{1/3} t^{2/3} \simeq 1.26 E^{1/3} t^{2/3}$. One should expect that the approximate solutions constructed by the method proposed above give satisfactory approximations in the neighborhood of $\tau = 0$. To take into account the dynamics of the process and the flow geometry, we consider a family of functions $g(\xi, \mathbf{T})$ with the vector

function $\mathbf{T}(\tau) = (T_1(\tau), \dots, T_k(\tau))$ and define the mappings $\mathbf{T} \rightarrow P(\xi, \mathbf{T})$, $\mathbf{T} \rightarrow V(\xi, \mathbf{T}, \tau \mathbf{T}', \tau)$ by equalities (3.11). This defines the Lagrangian

$$L = \sqrt{\tau} M(q, \mathbf{T}, \tau \mathbf{T}', \tau) + \frac{\sqrt{\tau}}{q} \left[E - \frac{e_0}{\tau(\nu+1)} (r^{\nu+1} - 1) \right],$$

where

$$M = \frac{1}{q} \int_0^1 r^\nu \left(\frac{q^2}{2} V^2 - P \right) d\xi.$$

Variation of the action functional with respect to q gives the energy integral (3.10), and variation of it with respect to \mathbf{T} leads to the following system of ordinary differential equations:

$$\frac{d}{d\tau} \frac{\partial M}{\partial T_i'} + \frac{1}{2\tau} \frac{\partial M}{\partial T_i'} = \frac{\partial M}{\partial T_i} \quad (i = 1, 2, \dots, k), \quad (3.17)$$

which degenerates at $\tau = 0$. Therefore, the initial data for this system must be determined in the process of solution. To do this, we use Taylor's formula $M = M_0(q, \mathbf{T}) + \tau \mathbf{N}(q, \mathbf{T}) \cdot \mathbf{T}' + O(\tau^2)$ and, similarly, we write the energy integral (3.10):

$$W_0(q, \mathbf{T}) + \tau W(q, \mathbf{T}) + \tau \mathbf{K}(q, \mathbf{T}) \cdot \mathbf{T}' = E + O(\tau^2). \quad (3.18)$$

Substituting representation (3.18) into the Euler equations (3.17), we obtain

$$\frac{3}{2} N_i - \frac{\partial M_0}{\partial T_i} + \tau \frac{d}{d\tau} N_i = O(\tau^2) \quad (i = 1, 2, \dots, k).$$

From the last equality and the energy integral it follows that the solution of system (3.17) is regular at $\tau = 0$ if

$$W_0(q, \mathbf{T}) = E, \quad \frac{3}{2} N_i(q, \mathbf{T}) = \frac{\partial}{\partial T_i} M_0(q, \mathbf{T}) \quad (i = 1, 2, \dots, k); \quad (3.19)$$

$$\frac{d}{d\tau} W_0(q, \mathbf{T}) + \mathbf{K}(q, \mathbf{T}) \cdot \mathbf{T}' + W(q, \mathbf{T}) = 0, \quad (3.20)$$

$$\frac{d}{d\tau} \left[\frac{3}{2} N_i(q, \mathbf{T}) - \frac{\partial}{\partial T_i} M_0(q, \mathbf{T}) + N_i(q, \mathbf{T}) \right] = 0 \quad (i = 1, 2, \dots, k)$$

at $\tau = 0$. System (3.19) defines the initial data for \mathbf{T} and q , and system (3.20) gives initial data for \mathbf{T}' [the value $q'(0)$ is calculated from the energy integral].

As a result, we obtain the following approximate formulation for the problem of pulsed compression of a finite mass of the mixture: *it is required to find a solution of the system of ordinary differential equations (3.17) with initial data defined by systems (3.19) and (3.20).*

Remark 15. For $\nu = 0$, problem (3.19) coincides with the problem of finding steady solutions from system (3.10) and (3.16). A solution of system (3.10) and (3.16) is a steady solution of the Euler equations (3.17) if $e_0 = 0$ [in this case, the energy integral is explicitly independent of τ and it follows from (3.20) that $q'(0) = 0$ and $\mathbf{T}'(0) = \mathbf{0}$].

As an example, we consider a family of approximate solutions in the following form: $V = T_1(\tau)\xi$ and $P = \exp\{T_2(\tau) + [T_1(\tau) - 1] \ln \xi\}$ for $\nu = 0$. The function M that defines the Lagrangian has the form

$$M = \frac{q}{54} [9(T_1 - \tau T_2')^2 + 24\tau T_1'(T_1 - \tau T_2') + 17\tau^2 T_2'^2] - \frac{1}{qT_1} e^{T_2}.$$

Another kind of approximate solutions is obtained under the assumption that $\tau_1 - \tau_0 \ll 1$ and, for $\tau \in (\tau_0, \tau_1)$, the velocity and pressure differ slightly from some averaged distributions $V(\xi)$ and $P(\xi)$. Therefore, let

$$P = \exp[r^{-\nu} q'(\xi)], \quad V = r^{-\nu} \left[q(\xi) - \xi q'(\xi) + \frac{1 - r^{\nu+1}}{\tau(\nu+1)} \right].$$

In this case, the continuity equation is satisfied.

Assuming that $V = 0$ at $\xi = 0$, from the equation in variations, we obtain

$$\begin{aligned}
 -\xi \frac{\partial}{\partial \xi} \int_{\tau_0}^{\tau_1} \sqrt{\tau} q V d\tau + \frac{\partial}{\partial \xi} \int_{\tau_0}^{\tau_1} \frac{\sqrt{\tau}}{q} P d\tau &= \int_{\tau_0}^{\tau_1} \sqrt{\tau} \left(\frac{1}{2} q + \frac{\nu \tau}{r} q \xi - \tau q' \right) V d\tau \\
 &= \int_{\tau_0}^{\tau_1} \frac{\sqrt{\tau}}{q} [P(1, \tau) - q^2 V(1, \tau)] d\tau = 0.
 \end{aligned}$$

Dividing these equalities by $\tau_1 - \tau_0$ and taking into account the continuity equation, in the limit $\tau_0, \tau_1 \rightarrow \tau$, we obtain the system

$$r^{-\nu} P (r^\nu V)_\xi = (\xi P)_\xi - \frac{\nu \tau \xi}{r} P \ln P; \quad (3.21)$$

$$P_\xi = q^2 \sqrt{\xi} (\sqrt{\xi} V)_\xi - \tau q (q' - \nu \xi q / r) V \quad (3.22)$$

with boundary conditions

$$P = q^2 V \quad (\xi = 1), \quad V = 0 \quad (\xi = 0). \quad (3.23)$$

Relation (3.21) is the continuity equation (3.1).

System (3.21)–(3.23) together with the energy integral (3.10) defines the approximate solutions of the problem of pulsed compression of a finite mass of the mixture with boundary conditions (3.4).

For $\nu = 0$ and $e_0 = 0$, the solution of the problem (3.21)–(3.23) is self-similar. Generally, the variable τ enters into the boundary-value problem as a parameter. The existence of the solution is not obvious.

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